THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra (Term 1, 2024-25)
Midterm Test Solutions
31st October, 2024

- Write your Name and Student ID on the front page.

- Give full explanation and justification for **<u>all</u>** your calculations and observations, and write all your proofs in a clear and rigorous way.

- Answer all questions. Total score: 100 pts. Time allowed: 90 minutes.

(1) (15 pts) Consider the subgroup

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R},\ ac \neq 0 \right\}$$

of $GL_2(\mathbb{R})$. (You can assume that this is indeed a subgroup of $GL_2(\mathbb{R})$ without proof.)

(a) Show that the subset

$$H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \subseteq G$$

is a normal subgroup of G. (Remember to check all the conditions!)

(b) Show that the quotient is isomorphic to $\mathbb{R}^\times \times \mathbb{R}^\times$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ is equipped with the multiplication of real numbers.

Answer.

(a) Let $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ be any elements in H, then $A^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ lies in H, and $AB = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$ also lies in H. So H is a subgroup.

Now for any $C \in G$, say $C = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, note that $C^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$, therefore $CAC^{-1} = \begin{pmatrix} 1 & \frac{ax}{c} \\ 0 & 1 \end{pmatrix} \in H$, so H is normal.

(b) Define a map $\varphi : G \to (\mathbb{R}^\times)^2$ by $\varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) := (a, c) \in \mathbb{R}^\times \times \mathbb{R}^\times$. Then φ is a homomorphism because

$$\begin{aligned}\varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} a\alpha & a\beta + b\gamma \\ 0 & c\gamma \end{pmatrix}\right) \\ &= (a\alpha, c\gamma) \\ &= (a, c) \times (\alpha, \gamma) \\ &= \varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) \times \varphi\left(\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\right).\end{aligned}$$

Also, φ is surjective because for any $a, c \in \mathbb{R}^\times \times \mathbb{R}^\times$, by definition it is the image $\varphi\left(\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\right)$.

The kernel is given by $\ker\varphi = \varphi^{-1}(1, 1)$ is precisely the set of matrices in G with diagonal entries given by 1, so $\ker\varphi = H$.

By the 1st isomorphism theorem, we have $G/H \cong \mathbb{R}^\times \times \mathbb{R}^\times$. ■

(2) (18 pts) List all possible isomorphism classes of abelian groups of order 540. Write your answers as direct products of finite cyclic groups.

Answer. There are 6 isomorphism classes:

(a) $\mathbb{Z}_{27} \times \mathbb{Z}_4 \times \mathbb{Z}_5$
(b) $\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
(c) $\mathbb{Z}_3^{\oplus 3} \times \mathbb{Z}_4 \times \mathbb{Z}_5$
(d) $\mathbb{Z}_{27} \times \mathbb{Z}_2^{\oplus 2} \times \mathbb{Z}_5$
(e) $\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_2^{\oplus 2} \times \mathbb{Z}_5$
(f) $\mathbb{Z}_3^{\oplus 3} \times \mathbb{Z}_2^{\oplus 2} \times \mathbb{Z}_5$

Note that there could be many equivalent ways to write down a group up to isomorphism. For example, $\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_{180} \cong \mathbb{Z}_9 \times \mathbb{Z}_{12} \times \mathbb{Z}_5$. ■

(3) (18 pts)

(a) Prove rigorously that a finite group always has a composition series.

(b) Give an example to show that a finite group may have two *different* composition series.

(c) State the Jordan-Holder Theorem. Explain why the example you gave in part (b) is not a counterexample to this uniqueness theorem.

Answer.

(a) Let G be a finite group. The poset (partially ordered set) of proper normal subgroups of G is finite, as it is a subposet of the poset of proper subsets of G, which is finite. Therefore it always contains a maximal element, i.e. a maximal proper normal subgroup.

Then we can prove that any finite G has a composition series by induction on $n = |G|$. If $n = 1$, it has a trivial composition series. Suppose for all groups with order $n \leq m - 1$ have a composition series, then for $|G| = m$, it has a maximal proper normal subgroup H. By induction hypothesis, H has a composition series

$$\{e\} = H_0 \lhd H_1 \lhd ... \lhd H_r \lhd H.$$

Then we can write down a composition series for G:

$$\{e\} = H_0 \lhd H_1 \lhd ... \lhd H \lhd G.$$

(b) Consider $\mathbb{Z}_6$, it has two composition series:

$$\{e\} \lhd \mathbb{Z}_2 \lhd \mathbb{Z}_6$$

and

$$\{e\} \lhd \mathbb{Z}_3 \lhd \mathbb{Z}_6$$

They are different composition series.

(c) Let G be a finite group, if

$$\{e\} = H_0 \lhd H_1 \lhd ... \lhd H_n = G$$

and

$$\{e\} = K_0 \lhd K_1 \lhd ... \lhd K_m = G$$

are two composition series for G, then $m = n$ and $\exists \sigma \in S_n$ such that

$$H_{i+1}/H_i \simeq K_{\sigma(i)+1}/K_{\sigma(i)}$$

There is no contradiction because the quotient groups of both composition series are $\mathbb{Z}_2$ and $\mathbb{Z}_3$. And they only differ by a permutation.

■

(4) (10 pts) Show that the direct product of finitely many solvable groups is solvable.

Answer. Notice that it suffices to prove that the direct product of two solvable groups is solvable, then one can proceed by induction to obtain the statement for the direct product of finitely many solvable groups. The inductive step is as follows: assume it is true that the direct product of k many solvable groups is solvable, then for the direct product of $k+1$ many solvable groups $A = G_1 \times ... \times G_{k+1}$, write $B = G_1 \times ... \times G_k$, then A is a direct product $A = (G_1 \times ... \times G_k) \times G_{k+1} = B \times G_{k+1}$. By induction hypothesis, $B = G_1 \times ... \times G_k$ is solvable, therefore $B \times G_{k+1}$ is also solvable.

Now it remains to prove the base case: the direct product of two groups is solvable. We give three proofs of this statement.

(First proof) Let G, H be solvable groups, by definition there exists subnormal series

$$G_0 \leq G_1 \leq ... \leq G_n = G$$

$$H_0 \leq H_1 \leq ... \leq H_m = H$$

such that G_i/G_{i-1} and H_j/H_{j-1} are abelian, for $i \in \{0, 1, ..., n\}$ and $j \in \{0, 1, ..., m\}$ respectively. Here to be precise, we used the notation that $G_{-1} = H_{-1} = \{e\}$. We will show that the subnormal series of $G \times H$ given by

$$G_0 \times \{e\} \leq G_0 \times H_0 \leq G_0 \times H_1 \leq ... \leq G_0 \times H_m \leq G_1 \times H_m \leq ... \leq G_n \times H_m = G \times H$$

has abelian quotient groups.

First of all, this is a subnormal series because direct product of normal subgroups is normal in the product group. Then we observe that the quotient groups either take the form $(G_0 \times H_j)/(G_0 \times H_{j-1})$ for some $j \in \{0, 1, ..., m\}$ or $(G_i \times H_m)/(G_{i-1} \times H_m)$ for some $i \in \{0, 1, ..., n\}$. In the first case, the quotient group is isomorphic to $(G_0/G_0) \times (H_j/H_{j-1}) \cong H_j/H_{j-1}$. In the second case, the quotient group is isomorphic to $(G_i/G_{i-1}) \times (H_m/H_m) \cong G_i/G_{i-1}$. In either cases, the quotient group is abelian. Therefore $G \times H$ is solvable.

(Second proof) We will use the criterion for solvability which says that a group K is solvable if and only if for a normal subgroup N of K, both N and K/N are solvable. For our purpose, assume that G, H are solvable groups, then $G \times \{e\} \leq G \times H$ is a normal subgroup since it is a product of normal subgroups of G and H. Note that both the normal subgroup $G \times \{e\} \cong G$ and the quotient $(G \times H)/(G \times \{e\}) \cong (G/G) \times (H/\{e\}) \cong H$ are solvable by assumption. Therefore by the criterion above, $G \times H$ is also solvable.

(Third proof) We will use another criterion for solvability which says that L is solvable if the derived series terminates at the trivial subgroup. Namely, we denote $L^{(1)} = [L, L]$, and $L^{(k)}$ defined inductively as $[L^{(k-1)}, L^{(k-1)}]$, then for the subnormal series $L^{(k)} \leq L^{(k-1)} \leq ... \leq L^{(1)} \leq L$, for large enough $k \in \mathbb{Z}_{>0}$, we have $L^{(k)} = \{e\}$. For our purpose, assume that G, H are solvable groups. Note that for arbitrary product groups $A \times B$, its derived subgroup $[A \times B, A \times B]$ is generated by all elements of the form $[(a_1, b_1), (a_2, b_2)] = (a_1, b_1) \cdot (a_2, b_2) \cdot (a_1, b_1)^{-1} \cdot (a_2, b_2)^{-1} = (a_1 a_2 a_1^{-1} a_2^{-1}, b_1 b_2 b_1^{-1} b_2^{-1}) = ([a_1, a_2], [b_1, b_2]) \in [A, A] \times [B, B]$. This computation immediately implies that $[A \times B, A \times$

$B] = [A, A] \times [B, B]$ (they are equal as subgroups, not only isomorphic). Therefore, the derived series of $G \times H$ would look like

$$G^{(k)} \times H^{(k)} \leq ... \leq G^{(1)} \times H^{(1)} \leq G \times H.$$

Since the derived series for G, H respectively would individually terminates at $\{e\}$, it follows that the derived series for $G \times H$ also terminates at $\{e\} \times \{e\}$ for sufficiently large k. ■

(5) (15 pts)

(a) Let G be a group and $H \leq G$ be a subgroup. Show that G acts on the set $L = \{aH : a \in G\}$ of left cosets of H in G by left multiplication, i.e., $g \cdot aH = (ga)H$.

(b) Show that the G-action in part (a) is faithful if and only if $\bigcap_{a \in G} aHa^{-1} = \{e\}$.

(c) Let X and Y be G-sets with the *same* group G. An **isomorphism** between the G-sets X and Y is a bijection $\phi : X \to Y$ which is **equivariant**, i.e., such that $\phi(g \cdot x) = g \cdot \phi(x)$ for all $x \in X$ and $g \in G$. Two G-sets are said to be **isomorphic** if there exists an equivariant bijection between them.

Let X be a transitive G-set, and let $x_0 \in X$. Show that X is isomorphic, as G-sets, to the G-set L of all left cosets of the stabilizer G_{x_0} in G defined in part (a).

Answer.

(a) Define the action $\rho : G \times L \to L$ via $\rho(g, aH) = g \cdot aH := (ga)H$. It is clear that the definition is independent of the choice of representative of the coset. Now we check that $e \cdot aH = (ea)H = aH$. And $g_1 \cdot (g_2 \cdot aH) = g_1 \cdot (g_2 a)H = (g_1 g_2 a)H = (g_1 g_2) \cdot aH$. Therefore ρ defines a G-action.

(b)

$$\begin{aligned}
G \text{ action on } L \text{ is faithful} &\iff g \in G \text{ acts trivially on } L \text{ if and only if } g = e \\
&\iff \{g \in G : \ g \cdot aH = aH, \forall aH \in L\} = \{e\} \\
&\iff \{g \in G : \ (a^{-1}ga)H = H, \forall a \in G\} = \{e\} \\
&\iff \{g \in G : \ a^{-1}ga \in H, \forall a \in G\} = \{e\} \\
&\iff \{g \in aHa^{-1} : \ \forall a \in G\} = \{e\} \\
&\iff \bigcap_{a \in G} aHa^{-1} = \{e\}.
\end{aligned}$$

(c) Fix $x_0 \in X$, by transitivity of the G-action on X, for any $x \in X$, there exists some $g \in G$ such that $x = g \cdot x_0$. Then we define $\phi : X \to L$ by $\phi(x) = gG_{x_0}$ for $g \in G$ satisfying $x = g \cdot x_0$. We have to show that ϕ is well-defined, i.e. it is independent on the choice of $g \in G$ in the above. If $g, h \in G$ both satisfy $x = g \cdot x_0 = h \cdot x_0$, then $x_0 = g^{-1} \cdot (g \cdot x_0) = g^{-1} \cdot (h \cdot x_0) = (g^{-1}h) \cdot x_0$. Therefore $g^{-1}h \in G_{x_0}$. This implies that $gG_{x_0} = hG_{x_0}$. So the function ϕ is well-defined.
Now for injectivity, if $\phi(x) = \phi(y)$, choose some $g_x, g_y \in G$ such that $x = g_x \cdot x_0$ and $y = g_y \cdot x_0$. Then $\phi(x) = g_x G_{x_0} = g_y G_{x_0} = \phi(y)$ implies that $g_x^{-1} g_y \in G_{x_0}$, therefore $g_x^{-1} g_y \cdot x_0 = x_0$ by definition. And we have $x = g_x \cdot x_0 = g_y \cdot x_0 = y$, as desired.
For surjectivity, we have for any $aH \in L$, it is the image $\phi(a \cdot x) = aH$.
For equivariance, let $g \in G$, and $x \in X$, choose some $g_x \in G$ such that $x = g_x \cdot x_0$. Then gg_x satisfies $gg_x \cdot x_0 = g \cdot (g_x \cdot x_0) = g \cdot x$. Therefore $\phi(g \cdot x) = gg_x G_{x_0} = g \cdot (g_x G_{x_0}) = g \cdot \phi(x)$. This concludes the proof. ■

(6) (24 pts) Let G be a finite simple group. Suppose that there exists a proper subgroup $H \lneq G$ such that $[G : H] \leq 4$.

(a) By considering the G-action on the set $X = \{aH : a \in G\}$ of left cosets of H in G by left multiplication (defined in part (a) of the previous question). Show that G is isomorphic to a subgroup of S_m where $m = [G : H] \leq 4$.

(b) Show that $|G| \leq 3$.

Answer.

(a) Consider the left action of G on the left cosets of H, then there is a group homomorphism $\rho : G \to S_m$, which is given by

$$\rho(g) = \rho_g,$$

where ρ_g is a permutation on X such that $\rho_g(xH) = (gx)H$. (We know that ρ defines a group action by the previous question). Since G is a simple group, $\ker \rho = \{e\}$. Then G is isomorphic to a subgroup of $S_{|X|} = S_m$ by the first isomorphism theorem.

(b) If $m = 1$ or $m = 2$, we are done. If $m = 3$, since G is a simple group, $G \neq S_3$. Hence $|G| \leq 3$.

If $m = 4$, all possible orders of G are $1, 2, 3, 4, 6, 8, 12, 24$. Since G is a simple group, $G \neq S_4$, $G \neq A_4$ because S_4 has a normal subgroup A_4 and A_4 has a normal subgroup $\langle (12)(34), (13)(24), (14)(23) \rangle$ which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since p-group always has non-trivial center and center must be normal subgroup, $|G| \neq 8$. $[G : H] = 4$ tells us $|G| \neq 6$. Again because G is simple, and all groups with order 4 never be simple, $|G| \leq 3$. ■